

# On the structure of the two-stream instability – complex G-Hamiltonian structure and Krein collisions between positive- and negative-action modes

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## Abstract

The two-stream instability is probably the most important elementary example of collective instabilities in plasma physics and beam-plasma systems. For a warm plasma with two charged particle species based on a 1D warm-fluid model, the instability diagram of the two-stream instability exhibits an interesting band structure that has not been explained. We show that the band structure for this instability is the consequence of the Hamiltonian nature of the warm two-fluid system. Interestingly, the Hamiltonian nature manifests as a complex G-Hamiltonian structure in wave-number space, which directly determines the instability diagram. Specifically, it is shown that the boundaries between the stable and unstable regions are locations for Krein collisions between eigenmodes with different Krein signatures. In terms of physics, this rigorously implies that the system is destabilized when a positive-action mode resonates with a negative-action mode, and that this is the only mechanism by which the system can be destabilized. It is anticipated that this physical mechanism of destabilization is valid for other collective instabilities in conservative systems in plasma physics, accelerator physics, and fluid dynamics systems, which admit infinite-dimensional Hamiltonian structures.

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## I. INTRODUCTION

The two-stream instability is a fundamental collective instability in plasma physics. In addition to its practical importance in many application areas [1–5], it has been studied as one of the most elementary example of plasma instabilities [6–8]. For a homogeneous cold plasma with two charged particle species drifting in the  $z$ -direction with different macroscopic velocities, the 1D linear electrostatic perturbations of the form  $\sim \exp(ikz - i\omega t)$  are unstable (exhibit exponential temporal growth) when the following well-known condition is satisfied [2],

$$0 < k^2 V^2 < (\omega_{p1}^{2/3} + \omega_{p2}^{2/3})^3, \quad (1)$$

where  $\omega_{pj}$  ( $j = 1, 2$ ) is the plasma frequency,  $v_j^0$  ( $j = 1, 2$ ) is the drift velocity of the  $j$ -th species, and  $V = v_2^0 - v_1^0$  is the relative drift velocity. Note that  $(\omega_{p1}^{2/3} + \omega_{p2}^{2/3})^{3/2}/k$  is the threshold value for the relative velocity, above which the mode is stabilized.

In this paper, we study the stability properties of the two-stream instability when the thermal velocities  $v_{Tj}$  ( $j = 1, 2$ ) of the two species are non-vanishing. The dispersion relation for two-stream excitations is given by [5]

$$\frac{\omega_{p1}^2}{(\omega - kv_1^0)^2 - k^2 v_{T1}^2} + \frac{\omega_{p2}^2}{(\omega - kv_2^0)^2 - k^2 v_{T2}^2} = 1, \quad (2)$$

which is derived from the warm two-fluid model in Sec. II. In the present study, we normalize  $(\omega, \omega_{pj}, kv_j^0, kV, kv_{Tj})$  by  $\omega_{p1}$ . In principle, the dispersion relation in Eq. (2) can be solved analytically. But it is straightforward to solve it numerically. One example is given in Fig. 1, where the instability diagram is presented in the parameter plane corresponding to  $kV$  and  $kv_{T2}$ , and other parameters are chosen to be  $\omega_{p1}^2 = 1$ ,  $\omega_{p2}^2 = 1836$ ,  $kv_1^0 = 0$ , and  $kv_{T1} = 1$ . In Fig. 1, the connected region between the upper curve and the lower curve is the unstable band, and the other two disconnected regions are the stable regions.

Comparing with the cold two-stream instability, the instability diagram for the warm two-stream instability is much more interesting. The unstable region is a connected band, the lower threshold for the relative velocity  $V$  is larger than zero, and there is no upper threshold for the relative velocity  $V$ . This is drastically different from the cold two-stream instability, which has an upper threshold  $(\omega_{p1}^{2/3} + \omega_{p2}^{2/3})^{3/2}/k$  for the relative velocity  $V$ . For the warm two-stream instability, for any value of  $V$ , about the lower threshold there is always a thermal velocity  $v_{T2}$  that destabilizes the mode. For a fixed relative velocity  $V$ , as  $v_{T2}$

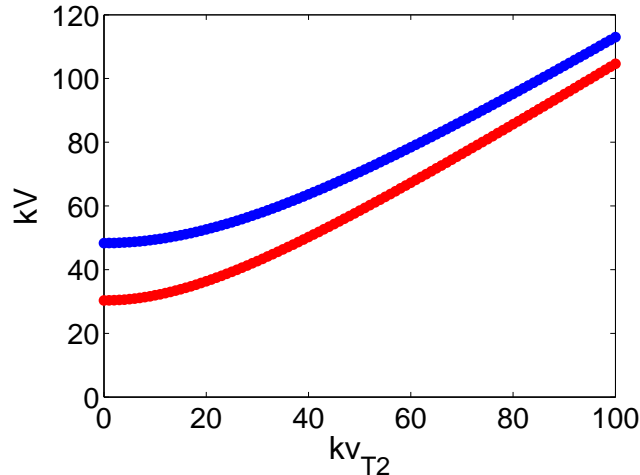


FIG. 1. Instability diagram for the warm two-stream instability in the parameter plane of  $kV$  and  $kv_{T2}$ . Other system parameters are chosen to be  $\omega_{p1}^2 = 1$ ,  $\omega_{p2}^2 = 1836$ ,  $kv_1^0 = 0$ , and  $kv_{T1} = 1$ . The connected region between the upper curve and the lower curve is the unstable band, and the other two disconnected regions are the stable regions.

increases from 0 to large values, the mode is at first stable, and then become unstable at a lower critical value, after which the mode is unstable until  $v_{T2}$  reaches an upper critical value. The modes becomes stable again after  $v_{T2}$  increases beyond the upper threshold. We observe a similar behavior when  $V$  varies for a fixed value of  $v_{T2}$ .

The purpose of the present study is to reveal the physical and mathematical structure of the band structure of the warm two-stream instability. It turns out, in term of the key physics, that the band structure is largely due to the conservative Hamiltonian nature of the two-stream interaction. And the mathematical description of the Hamiltonian nature is expressed as a complex G-Hamiltonian structure, which is a complex, non-canonical generalization of the familiar real canonical Hamiltonian structure. The mathematical theory of the linear G-Hamiltonian system has been systematically developed by Krein, Gel'fand and Lidskii [9–11]. As the system parameters vary, a necessary and sufficient condition for the onset of instability is that two eigenmodes with opposite Krein signatures collide (see Sec.III). This is the so-called Krein collision. We will show that the physical meaning of the Krein signature is the sign of the action for the eigenmode. For a stable mode, the action is defined to be the ratio between the energy and the frequency of the eigenmode. We will show that the instability boundaries in Fig.1 are the locations for the Krein collisions.

In terms of essential physics, the instability boundaries are located where a positive-action mode resonates with a negative-action mode.

To the best of our knowledge, the study present in this paper is the first of its kind to systematically apply the G-Hamiltonian theoretical framework to plasma physics applications. However, we would like to mention a few similar concepts and methods that have been discussed by the plasma physics community. The first concept is the Hamiltonian-Hopf bifurcation. To those who are familiar with Hamiltonian dynamics, the boundary between a stable and unstable equilibrium is known as the Hamiltonian-Hopf bifurcation. One may wonder whether the instability boundaries in Fig. 1 are Hamiltonian-Hopf bifurcations. The answer is no. This is because the Hamiltonian-Hopf bifurcation is specifically for a real Hamiltonian system, and our dynamical system in time is a complex G-Hamiltonian system for one Fourier component in space. To a certain extent, the Krein collision at the instability boundaries in Fig. 1 can be viewed as a complex generalization of the Hamiltonian-Hopf bifurcation. Since all dynamical systems in plasmas are complex after Fourier decomposition in space, we expect that this feature applies to all plasma instabilities in infinite dimensional Hamiltonian systems, i.e., an unstable mode is created and can only be created by the resonance between two stable modes with opposite-sign actions.

Another related concept is that of the negative- and positive-energy modes [6–8, 12]. Such concepts have appeared from time to time in the plasma physics literature, mostly in connection with the destabilization of certain modes. It has been suggested that instability occurs when a negative-energy mode resonates with a positive-energy mode. However, this process has only been discussed at an intuitive level without a rigorous mathematical description. What we show in the present study is that the relevant concept is not that of negative or positive energy. Instead, what matters is whether the action is negative or positive. We will provide a rigorous definition of the action of an eigenmode, and use the example of the warm two-stream instability to show mathematically that system becomes unstable when and only when a positive-action mode resonates with a negative-action mode.

The Hamiltonian structure of the linear two-stream equation system for a single  $k$  was first studied by Kueny and Morrison [6, 7]. When the system is stable, a set of coordinate transformations were given to transform the complex system into an 8-D real canonical normal form, which applies to cases with two distinct real eigenfrequencies. In the general case, the dispersion relation (2) for the warm two-stream modes admits four distinct complex

eigenfrequencies. Particularly noteworthy, both the real and imaginary parts of an eigenfrequency can be non-vanishing (see Figs. 2 and 3). The complex G-Hamiltonian structure for the warm two-stream system discovered in the present study applies to the most general warm two-stream modes. This should not be surprising. As a matter of fact, the only symmetry constraint that the dispersion relation (2) imposes is that the eigenfrequencies are symmetric with respect to the real axis, and this is also the only symmetry constraint that a G-Hamiltonian matrix admits (see Theorem 1). This is why a  $4 \times 4$  complex G-Hamiltonian structure is able to faithfully represent the general spectrum of the warm two-stream system. On the other hand, we can attempt to use a real symplectic system to represent the spectrum of the warm two-stream system. However, we note that the eigenfrequencies of a  $sp(2n)$  matrix are symmetric with respect to both the real and imaginary axes, and the constraint of being symmetric with respect to the imaginary axis is not that of the warm two-fluid system. Therefore, one has to use a real symplectic system whose dimension is at least 16. Here,  $sp(2n)$  is the Lie algebra of the Lie group of symplectic matrices  $Sp(2n)$ .

In plasma physics and accelerator physics, there are many other applications, e.g., charged particle dynamics in a periodic focusing lattice [13–15], where the underlying systems are finite-dimensional real Hamiltonian systems. In these cases, the Krein collision theory is also valid for these real symplectic systems. Stability analyses using the Krein collision theory have been successfully applied to these systems to generate results of practical importance [16–18] .

This paper is organized as follows. In Sec. II, the dispersion relation for the warm two-stream instability is derived from a set of linearized warm two-fluid equations. In Sec. III the theory of G-Hamiltonian systems and Krein collisions are introduced. We then show that the warm two-stream system in wave-number space is a G-Hamiltonian system in Sec. IV, and analyze the structure of the instability diagram using the physical mechanism of resonance between positive- and negative-action modes.

## II. THEORETICAL MODEL FOR WARM TWO-STREAM INSTABILITY

We consider the warm two-fluid system describing a non-relativistic plasma with 1D spatial variations and electrostatic fields in the  $z$ -direction,

$$\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial z}(n_j v_j) = 0, \quad (3)$$

$$\frac{\partial v_j}{\partial t} + v_j \frac{\partial v_j}{\partial z} + \frac{1}{n_j m_j} \frac{\partial p_j}{\partial z} = \frac{e_j}{m_j} E, \quad (4)$$

$$\frac{\partial E}{\partial z} = \sum_j 4\pi e_j n_j, \quad (5)$$

$$p_j = \frac{p_{j0}}{\hat{n}_j^{\gamma_j}} n_j^{\gamma_j}, \quad (6)$$

where  $j = 1, 2$  is the index labeling plasma species,  $e_j$  and  $m_j$  are the charge and mass of a particle of species  $j$ ,  $\gamma_j$  is the polytropic index, and  $p_{j0}$  and  $\hat{n}_j$  are constants. Considering small perturbations  $\tilde{\rho}_j = e_j \tilde{n}_j$  and  $\tilde{v}_j$  about the constant values  $e_j \hat{n}_j$  and  $v_j^0$ , we obtain the linearized fluid equations in terms of  $\tilde{\rho}_j$ ,

$$\left( \frac{\partial}{\partial t} + v_1^0 \right) \left( \frac{\partial}{\partial t} + v_1^0 \right) \tilde{\rho}_1 - v_{T1}^2 \tilde{\rho}_1 = -\omega_{p1}^2 (\tilde{\rho}_1 + \tilde{\rho}_2), \quad (7)$$

$$\left( \frac{\partial}{\partial t} + v_2^0 \right) \left( \frac{\partial}{\partial t} + v_2^0 \right) \tilde{\rho}_2 - v_{T2}^2 \tilde{\rho}_2 = -\omega_{p2}^2 (\tilde{\rho}_1 + \tilde{\rho}_2), \quad (8)$$

where  $\omega_{pj}^2 = 4\pi \hat{n}_j e_j^2 / m_j$  and  $v_{Tj}^2 = \gamma_j T_j / m_j$ . For a single Fourier mode in space  $\tilde{\rho}_j \sim \exp(ikz)$ , where  $k$  is the wavenumber of the perturbation, Eqs. (7) and (8) reduce to two coupled ordinary differential equations

$$\left( \frac{d}{dt} + ikv_1^0 \right)^2 \tilde{\rho}_1 + k^2 v_{T1}^2 \tilde{\rho}_1 = -\omega_{p1}^2 (\tilde{\rho}_1 + \tilde{\rho}_2), \quad (9)$$

$$\left( \frac{d}{dt} + ikv_2^0 \right)^2 \tilde{\rho}_2 + k^2 v_{T2}^2 \tilde{\rho}_2 = -\omega_{p2}^2 (\tilde{\rho}_1 + \tilde{\rho}_2), \quad (10)$$

which can also be expressed in the compact form of a 4-dimensional linear complex dynamical system,

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad (11)$$

where

$$A = \begin{pmatrix} -2ikv_1^0 & 0 & (kv_1^0)^2 - \omega_{p1}^2 - k^2v_{T1}^2 & -\omega_{p1}^2 \\ 0 & -2ikv_2^0 & -\omega_{p2}^2 & (kv_2^0)^2 - \omega_{p2}^2 - k^2v_{T2}^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (12)$$

$$\mathbf{x} = \begin{pmatrix} d\tilde{\rho}_1/dt \\ d\tilde{\rho}_2/dt \\ \tilde{\rho}_1 \\ \tilde{\rho}_2 \end{pmatrix}. \quad (13)$$

Starting from Eq. (11), if we take  $\mathbf{x} \sim \exp(-i\omega t)$ , then  $\lambda = -i\omega$  will be the eigenvalues of the complex matrix  $A$ , and the eigenvalues are determined by the eigen-polynomial of  $A$ . Note that an eigenvalue  $\lambda$  and an eigenfrequency  $\omega$  are two different quantities. But they are connected by the simple relation,  $\omega = i\lambda$ . In terms of the eigenfrequency  $\omega$ , the eigen-polynomial of  $A$  is the dispersion relation (2). The instability diagram determined from the dispersion relation is shown in Fig. 1. As discussed in Sec. I, we will focus on the complex G-Hamiltonian nature of Eq. (11), and reveal the fundamental connection between the G-Hamiltonian nature of Eq. (11) and the instability structure in Fig. 1.

### III. G-HAMILTONIAN SYSTEM AND KREIN'S THEORY

A complex Hamiltonian system [19] in the space of  $(\mathbf{z}, \bar{\mathbf{z}})$ , where  $\mathbf{z} \in C^n$  and  $\bar{\mathbf{z}} \in C^n$ , is defined by the following Poisson bracket between two functions  $f(z, \bar{z})$  and  $g(z, \bar{z})$ ,

$$\{f, g\} = \frac{1}{i} \sum_j \left( \frac{\partial f}{\partial \mathbf{z}_j} \frac{\partial g}{\partial \bar{\mathbf{z}}_j} - \frac{\partial f}{\partial \bar{\mathbf{z}}_j} \frac{\partial g}{\partial \mathbf{z}_j} \right), \quad (14)$$

and a Hamiltonian function  $H(z, \bar{z})$ . The functions  $f(z, \bar{z})$ ,  $g(z, \bar{z})$ , and  $H(z, \bar{z})$  are complex functions from  $C^n \times C^n$  to  $C$ . The dynamical equations for  $\mathbf{z} \in C^n$  and  $\bar{\mathbf{z}} \in C^n$  are given by

$$\dot{\mathbf{z}} = \{\mathbf{z}, H\}, \quad (15)$$

$$\dot{\bar{\mathbf{z}}} = \{\bar{\mathbf{z}}, H\}, \quad (16)$$

which are explicitly

$$\dot{\mathbf{z}} = \frac{1}{i} \frac{\partial H}{\partial \bar{\mathbf{z}}}, \quad (17)$$

$$\dot{\bar{\mathbf{z}}} = -\frac{1}{i} \frac{\partial H}{\partial \mathbf{z}}. \quad (18)$$

If we require the Hamiltonian function to satisfy the reality condition, i.e.,  $H(\mathbf{z}, \bar{\mathbf{z}}) : C^n \times C^n \rightarrow R$  or

$$H(\mathbf{z}, \bar{\mathbf{z}}) = \overline{H(\mathbf{z}, \bar{\mathbf{z}})}, \quad (19)$$

then it follows that

$$\frac{\partial H}{\partial \bar{\mathbf{z}}} = \overline{\frac{\partial H}{\partial \mathbf{z}}}. \quad (20)$$

To show this, we re-express the Hamiltonian function as  $H(\mathbf{z}, \bar{\mathbf{z}}) = H'(\mathbf{q}(\mathbf{z}, \bar{\mathbf{z}}), \mathbf{p}(\mathbf{z}, \bar{\mathbf{z}})) \in R$ , where  $\mathbf{z} = \frac{\mathbf{q} + i\mathbf{p}}{\sqrt{2}}$ , and  $\mathbf{q}$  and  $\mathbf{p}$  are real. By the chain rule,

$$\frac{\partial H}{\partial \bar{\mathbf{z}}} = \frac{\partial H'}{\partial \mathbf{q}} \frac{1}{\sqrt{2}} + \frac{\partial H'}{\partial \mathbf{p}} \left(-\frac{1}{\sqrt{2}i}\right), \quad (21)$$

$$\frac{\partial H}{\partial \mathbf{z}} = \frac{\partial H'}{\partial \mathbf{q}} \frac{1}{\sqrt{2}} + \frac{\partial H'}{\partial \mathbf{p}} \left(\frac{1}{\sqrt{2}i}\right), \quad (22)$$

which implies that Eq. (20) holds. As a consequence, Eqs. (17) and (18) are equivalent. In this case, the complex Hamiltonian system, i.e., Eqs. (17) and (18), are equivalent to the real canonical Hamiltonian system in terms of  $\mathbf{q}$  and  $\mathbf{p}$  [20],

$$\begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \partial H' / \partial \mathbf{q} \\ \partial H' / \partial \mathbf{p} \end{pmatrix}. \quad (23)$$

In the present study, we define a general complex G-Hamiltonian system for  $\mathbf{z} \in C^n$  and  $\bar{\mathbf{z}} \in C^n$  to be

$$\dot{\mathbf{z}} = \frac{1}{i} G^{-1} \frac{\partial H}{\partial \bar{\mathbf{z}}}, \quad (24)$$

$$\dot{\bar{\mathbf{z}}} = -\frac{1}{i} \bar{G}^{-1} \frac{\partial H}{\partial \mathbf{z}}, \quad (25)$$

where  $G$  is a non-singular Hermite matrix and the Hamiltonian function  $H(\mathbf{z}, \bar{\mathbf{z}})$  satisfies the reality condition in Eq. (19). Because of Eq. (20), Eqs. (24) and (25) are equivalent, and we only need to investigate one of them.



For a linear G-Hamiltonian systems satisfying the reality condition, its Hamiltonian function may assume the form of

$$H(\mathbf{z}, \bar{\mathbf{z}}) = \mathbf{z}^T M \mathbf{z} + \bar{\mathbf{z}}^T \bar{M} \bar{\mathbf{z}} + \bar{\mathbf{z}}^T S \mathbf{z}, \quad (26)$$

where  $M$  is a symmetric matrix and  $S$  is a Hermite matrix. In the present study, we will focus on a special class of linear G-Hamiltonian system in the form of

$$\dot{\mathbf{x}} = A \mathbf{x}, \quad (27)$$

$$A = iG^{-1}S, \quad (28)$$

where  $G$  is a non-singular Hermite matrix and  $S$  is a Hermite matrix. The corresponding Hamiltonian function is

$$H(\mathbf{x}) = -\mathbf{x}^* S \mathbf{x}. \quad (29)$$

This special linear G-Hamiltonian system is called Hamiltonian system and has been studied in detail by Yakubovich and Starzhinskii [11] without the Hamiltonian structure, i.e., Eq. (24) or (25), and the Hamiltonian function (29). However, we will see in Sec. IV that the Hamiltonian structure and Hamiltonian function will bring important physical insight to the mathematical theory of Krein collisions. Since in this paper we will not discuss nonlinear complex G-Hamiltonian systems and general linear G-Hamiltonian systems specified by Eq. (26), the G-Hamiltonian system in the rest of this section and other sections refers to the special linear G-Hamiltonian system specified by Eqs. (27)-(29).

A matrix  $A$  that can be expressed in the form of Eq. (28) is call a G-Hamiltonian matrix by Yakubovich and Starzhinskii [11], and we will follow this convention. An equivalent condition for  $A$  to be G-Hamiltonian is that there exists a nonsingular Hermite matrix  $G$  such that

$$A^* G + G A = 0. \quad (30)$$

For any two vectors  $\psi$  and  $\phi$ , a product is defined as

$$\langle \psi, \phi \rangle = \phi^* G \psi. \quad (31)$$

Following Yakubovich and Starzhinskii [11], we categorize the eigenvalues of a G-Hamiltonian matrix  $A$  according to their Krein signatures as follows:

- (a) For a simple eigenvalue  $\lambda$  of  $A$  on the imaginary axis, i.e.,  $Re(\lambda) = 0$ , let the corresponding eigenvector be  $\mathbf{y}$ . It can be shown that  $\langle \mathbf{y}, \mathbf{y} \rangle \neq 0$ . Thus, we define the eigenvalue  $\lambda$  to be the first kind if  $\langle \mathbf{y}, \mathbf{y} \rangle > 0$ , and the second kind if  $\langle \mathbf{y}, \mathbf{y} \rangle < 0$ .
- (b) For an  $r$ -fold eigenvalue  $\lambda$  of  $A$  on the imaginary axis, let  $V_\lambda$  be its eigen-subspace. If  $\langle \mathbf{y}, \mathbf{y} \rangle > 0$  for any  $\mathbf{y} \in V_\lambda$ , then  $\lambda$  is defined to be the first kind. If  $\langle \mathbf{y}, \mathbf{y} \rangle < 0$  for any  $\mathbf{y} \in V_\lambda$ , then eigenvalue  $\lambda$  is defined to be the second kind. This category includes (a) as a special case when  $r = 1$ .
- (c) For an  $r$ -fold eigenvalue  $\lambda$  of  $A$  on the imaginary axis, let  $V_\lambda$  be its eigen-subspace. If  $\langle \mathbf{y}, \mathbf{y} \rangle = 0$  for a  $\mathbf{y} \in V_\lambda$ , then eigenvalue  $\lambda$  is defined to be the mixed kind. An equivalent condition for a mixed kind is that there exist two eigen-vectors  $\mathbf{y}_1 \in V_\lambda$  and  $\mathbf{y}_2 \in V_\lambda$ , such that  $\langle \mathbf{y}_1, \mathbf{y}_1 \rangle > 0$  and  $\langle \mathbf{y}_2, \mathbf{y}_2 \rangle < 0$ .
- (d) For an  $r$ -fold eigenvalue  $\lambda$  of  $A$  not on the imaginary axis, i.e.,  $Re(\lambda) \neq 0$ , the eigenvalue  $\lambda$  is defined to be the first kind if  $Re(\lambda) < 0$ , and the second kind if  $Re(\lambda) > 0$ .

The first kind is assigned a Krein signature of  $+$ , the second kind is assigned a Krein signature of  $-$ , and the mixed kind is assigned a Krein signature of  $0$ . The first kind and the second kind are also called definite, and the mixed kind is also called indefinite.

Without giving proofs, we list the following theorems regarding the properties of the eigenvalues of a G-Hamiltonian matrix that will be used in the present study. The proofs can be found in Ref. [11].

**Theorem 1.** *The eigenvalues of a G-Hamiltonian matrix are symmetric with respect to the imaginary axis.*

**Theorem 2.** *The number of each kind of eigenvalue is determined by the Hermite matrix  $G$ . Let  $p$  be the number of positive eigenvalues and  $q$  be the number of negative eigenvalues of the matrix  $G$ , then any G-Hamiltonian matrix has  $p$  eigenvalues of first kind and  $q$  eigenvalues of second kind (counting multiplicity).*

**Theorem 3.** *(Krein-Gel'fand-Lidskii theorem) The G-Hamiltonian system (27) is strongly stable if and only if all of the eigenvalues of  $A$  lie on the imaginary axis and are definite.*

By definition, a G-Hamiltonian system (27) is strongly stable if all systems nearby are also stable. Systems nearby include those with system parameters perturbed by an infinitesimal

amount relative to the original system. The Krein-Gel'fand-Lidskii theorem tells us that for a stable G-Hamiltonina system, the only route for the system to become unstable when varying the system parameters is through the overlap between two eigenvalues with different Krein signatures on the imaginary axis. Such overlaps are called Krein collisions. Once one of the eigenvalues is “knocked off” the imaginary axis, the system must be unstable, because according to Theorem 1 the eigenvalues are symmetric with respect to the imaginary axis.

In the next section will we show that the warm two-stream system does have a G-Hamiltonian structure, and thus its instability is governed by the Krein collision process.

#### IV. G-HAMILTONIAN STRUCTURE AND KREIN COLLISION BETWEEN POSITIVE- AND NEGATIVE- ACTION MODES

It is found that the linear system for the warm two-stream dynamics described by Eq. (11) has a G-Hamiltonian structure, because

$$A = \begin{pmatrix} -2ikv_1^0 & 0 & (kv_1^0)^2 - \omega_{p1}^2 - k^2v_{T1}^2 & -\omega_{p1}^2 \\ 0 & -2ikv_2^0 & -\omega_{p2}^2 & (kv_2^0)^2 - \omega_{p2}^2 - k^2v_{T2}^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = iG^{-1}S, \quad (32)$$

where

$$G = \begin{pmatrix} 0 & 0 & i\omega_{p2}^2 & 0 \\ 0 & 0 & 0 & i\omega_{p1}^2 \\ -i\omega_{p2}^2 & 0 & 2kv_1^0\omega_{p2}^2 & 0 \\ 0 & -i\omega_{p1}^2 & 0 & 2kv_2^0\omega_{p1}^2 \end{pmatrix}, \quad (33)$$

and

$$S = \begin{pmatrix} \omega_{p2}^2 & 0 & 0 & 0 \\ 0 & \omega_{p1}^2 & 0 & 0 \\ 0 & 0 & -\omega_{p2}^2((kv_1^0)^2 - \omega_{p1}^2 - k^2v_{T1}^2) & \omega_{p1}^2\omega_{p2}^2 \\ 0 & 0 & \omega_{p1}^2\omega_{p2}^2 & -\omega_{p1}^2((kv_2^0)^2 - \omega_{p2}^2 - k^2v_{T2}^2) \end{pmatrix}. \quad (34)$$

Equation (32) can be verified by direct calculation. According to Theorem 2, the distribution of Krein signatures of the eigenvalues for  $A$  is determined by  $G$  given by Eq. (33). It has

four eigenvalues,

$$\begin{aligned} & kv_2^0 \omega_{p1}^2 - \sqrt{(1 + (kv_2^0)^2) \omega_{p1}^4}, \quad kv_2^0 \omega_{p1}^2 + \sqrt{(1 + (kv_2^0)^2) \omega_{p1}^4}, \\ & kv_1^0 \omega_{p2}^2 - \sqrt{(1 + (kv_1^0)^2) \omega_{p2}^4}, \quad kv_1^0 \omega_{p2}^2 + \sqrt{(1 + (kv_1^0)^2) \omega_{p2}^4} \end{aligned} \quad (35)$$

Two of them are positive and the other two are negative. Thus, the G-Hamiltonian matrix  $A$  has two eigenvalues of the first kind and two eigenvalues of the second kind.

We now show that the band structure of the instability region in Fig. 1 is produced by the Krein collision process. But, we will first explain the physical meaning of the Krein signature. For an eigenvalue  $\lambda = -i\omega$  of  $A$  on the imaginary axis with an eigenvector  $\mathbf{y}$ , it follows that

$$iG^{-1}S\mathbf{y} = A\mathbf{y} = -i\omega\mathbf{y}. \quad (36)$$

Because of Eqs. (29) and (31), we obtain

$$\langle \mathbf{y}, \mathbf{y} \rangle = \frac{H(\mathbf{y})}{\omega}. \quad (37)$$

It is clear that for an eigenvalue  $\lambda$  of  $A$  on the imaginary axis, the physical meaning of its signature is the sign of the action, which is defined to be the ratio between energy and the eigenfrequency of the mode. This gives us the following physical interpretation of the celebrated Krein-Gel'fand-Lidskii theorem: the system becomes unstable when and only when a negative-action mode resonates with a positive-action mode. We emphasize that it is not accurate to state that the system becomes unstable when and/or only when a negative-energy mode resonates with a positive-energy mode, because two positive-energy modes (or two negative-energy modes) can collide at zero frequency to destabilize the system. For an eigenvalue  $\lambda$  of  $A$  that is not on the imaginary axis, its action is defined to be positive if  $Im(\omega) < 0$ , and negative if  $Im(\omega) > 0$ .

Now we show that band of instability region in Fig. 1 is indeed the result of a Krein collision. In Fig. 2, we vary the value of  $kv_{T2}$  and fix all other system parameters at  $\omega_{p1}^2 = 1$ ,  $\omega_{p2}^2 = 1836$ ,  $kv_1^0 = 0$ ,  $kv_2^0 = 50$ , and  $kv_{T1} = 1$ . This motion corresponds to traveling horizontally in Fig. 1 at  $kv_2^0 = 50$ . When  $kv_{T2} = 10$ , the eigenvalues of  $A$  are all on the imaginary axis and are distinct, and the system is stable. Two of the eigenmodes (marked by red) have positive actions and the other two (marked by green) have negative actions. As  $kv_{T2}$  increases, one of the positive-action mode moves towards one of the negative-action mode. At  $kv_{T2} = 12.2834$ , the two modes with opposite-sign actions collide and

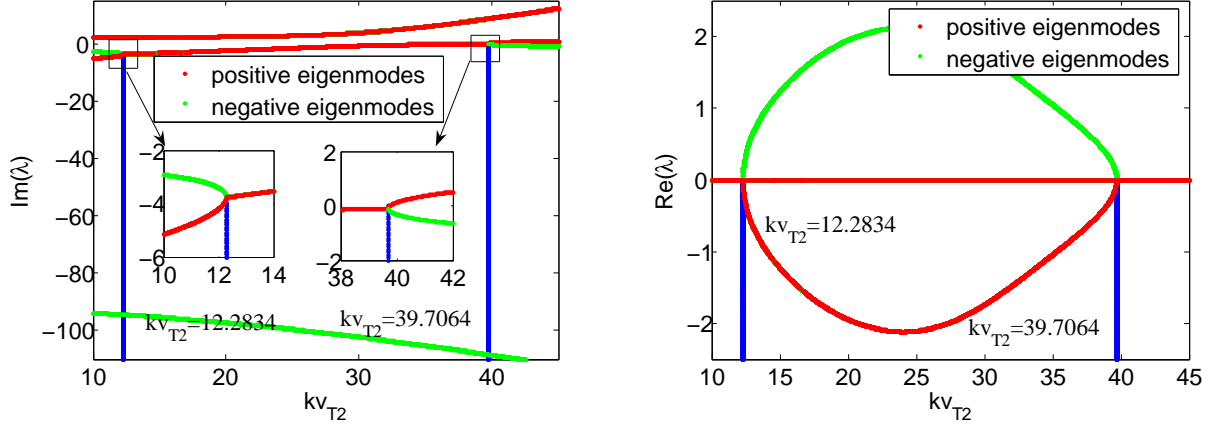


FIG. 2. Plots of Krein collisions when varying  $kv_{T2}$  at  $w_{p1}^2 = 1$ ,  $w_{p2}^2 = 1836$ ,  $kv_1^0 = 0$ ,  $kv_2^{60} = 50$  and  $kv_{T1} = 1$ . Krein collisions occur at  $kv_{T2} = 12.2834$  and  $kv_{T2} = 39.7064$ .

destabilize the system. This point defines the upper instability boundary in Fig. 1. When  $kv_{T2}$  increases beyond this threshold, these two eigenvalues move off the imaginary axis, and the system is unstable. Because the eigenvalues are symmetric with respect to the imaginary axis according to Theorem 1, these two complex eigenvalues must have the same imaginary parts, but opposite real parts. Increasing  $kv_{T2}$  to 39.7064 will strike the lower instability boundary in Fig. 1. This corresponds to another Krein collision between two modes with opposite-sign actions. When  $kv_{T2} > 39.7064$ , the system will again have four distinct eigenvalues on the imaginary axis, and the system is stable. There are two ways to look at the Krein collision at  $kv_{T2} = 39.7064$ . In the direction of increasing  $kv_{T2}$ , we observe a Krein collision between two eigenmodes with opposite-sign actions in an unstable system, and the system is stabilized by the collision. In the direction of decreasing  $kv_{T2}$ , we observe a Krein collision between two stable eigenmodes with opposite-sign actions, and the system is destabilized by the collision. Of course, this point-of-view applies to the Krein collision at  $kv_{T2} = 12.2834$  as well.

In Fig. 3, we vary  $kv_2^0$  while fixing all other parameters at  $\omega_{p1}^2 = 1$ ,  $\omega_{p2}^2 = 1836$ ,  $kv_1^0 = 0$ ,  $kv_{T1} = 1$ , and  $kv_{T2} = 5$ . This corresponds to traveling vertically in Fig. 1 at  $kv_{T2} = 5$ . The dynamics of Krein collisions in this motion are similar to the dynamics in Fig. 2, i.e., the upper and lower instability boundaries in this motion correspond to Krein collisions between two eigenmodes with actions that have opposite signs.

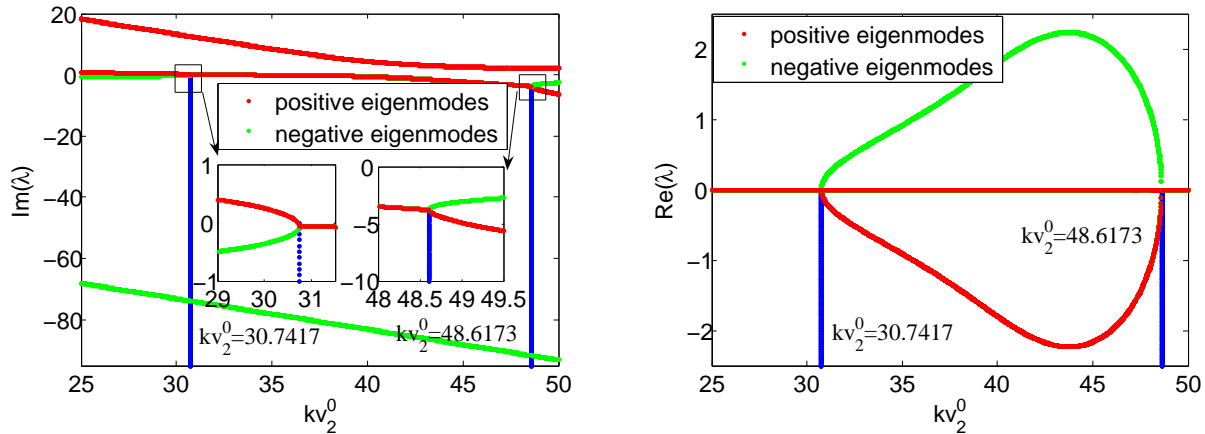


FIG. 3. Plots of Krein collisions when varying  $kv_2^0$  at  $w_{p1}^2 = 1$ ,  $w_{p2}^2 = 1836$ ,  $kv_1^0 = 0$ ,  $kv_{T1} = 1$  and  $kv_{T2} = 5$ . Krein collisions occur at  $kv_2^0 = 30.7417$  and  $kv_2^0 = 48.6173$ .

Finally, we describe an interesting phenomenon. We ask the question whether the system is stable when it is exactly on the boundaries between the stable and unstable regions, i.e., at the points of the Krein collisions. From the eigenvalue point of view, the eigenvalues are on the imaginary axis (or the eigenfrequencies are on the real axis), one may expect that the system is stable. This is not correct, because on these boundaries the eigenvalues are repeated eigenvalues. Whether the system is stable or not depends on whether matrix  $A$  can be diagonalized or not. For the warm two-stream system studied here, it turns out that there is only one Jordan block for the repeated eigenfrequencies, and the system is algebraically unstable, i.e., the perturbation grows linearly with time.

## V. CONCLUSIONS

In this paper, we have shown that the dynamics of warm two-stream modes in wave-number space are governed by a complex G-Hamiltonian structure, which directly determines the structure of the instability diagram. We have rigorously shown that the system is destabilized when and only when a positive action mode resonates with a negative action mode. It is anticipated that this physical picture of the G-Hamiltonian structure and destabilization mechanism by resonances between two modes with actions that have opposite signs holds for other collective instabilities in conservative systems in plasma physics, accelerator

physics, and fluid dynamics that admit infinite-dimensional Hamiltonian structures.

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